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Non-perturbative gauge superpotentials from supergravity

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Abstract: We study U(N) SQCD with $N_f \leq N$ flavors of quarks and antiquarks by engineering it with a configuration of fractional D3-branes on a $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. In particular we show how the moduli space of the gauge theory naturally emerges from the classical geometry produced by the D3-branes, and how the non-perturbatively generated superpotential is recovered from geometrical data.

Keywords: D-branes, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

The search for extensions of the gauge/gravity correspondence towards non-conformal theories with a reduced number of supercharges has recently achieved some interesting progress. Indeed, it has been shown that many relevant properties of non-conformal $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions can be extracted from dual supergravity solutions associated to different set-ups of strings and D-branes. For example, at the perturbative level one can get the correct logarithmic running of the coupling constant and the chiral anomaly, while at the non-perturbative level one can discuss instantons, gaugino condensation and confining strings.

Many of these results can be obtained by studying in detail the supergravity solutions produced by stacks of fractional D3-branes in conifold [1]–[5] and orbifold [1] and [6]–[8] backgrounds. Alternatively, non-conformal gauge theories with $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetry can be realized by wrapping D-branes on suitable supersymmetric cycles in Calabi-Yau or K3 manifolds [9]. Also in these cases, a detailed study of the corresponding supergravity solutions has provided relevant information on the dual gauge theory [4, 10]. Most of these developments are covered in recent reviews [11, 12] to which we refer also for a more extended bibliography.

Another line of research has been the study of the so-called "geometric transitions" [13, 14], where one engineers gauge theories by wrapping D5-branes on two-cycles of resolved Calabi-Yau manifolds in such a way that the geometry flows to a deformed manifold where branes are replaced by fluxes through the newly blown-up three-cycles. In this framework, it has been shown [15, 16] how to get the non-perturbatively generated effective superpotential of the dual $\mathcal{N} = 1$ gauge theory [17, 14] by means of geometric considerations.

In this paper we bring together these lines of research, and show that the explicit knowledge of the supergravity solution produced by a stack of D-branes can be fruitfully combined with general geometric considerations on the background in which they are embedded. In this way one can obtain relevant information on the dual gauge theory, both at a perturbative and at a non-perturbative level. In particular we will discuss the Affleck-

Dine-Seiberg theory, namely U(N) $\mathcal{N}=1$ Super QCD with $N_f < N$ flavors of quarks and antiquarks, which we realize as the world-volume theory of a stack of fractional D3-branes on a $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. We then show how to use this fractional brane configuration to obtain the running coupling constant, the classical moduli space of the low-energy theory and the non-perturbative effective superpotential [18] of SQCD in the chosen regime. We also comment on SQCD with $N = N_f$ which simply arises as a particular case of our construction.

This paper is organized as follows. In section 2 we introduce the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and show how U(N) SQCD can be engineered via a specific configuration of fractional D3-branes. After summarizing in section 3 the corresponding supergravity solution, in section 4 we study the dual gauge theory, deriving the running gauge coupling constant and exploring the classical moduli space. Finally, in section 5 we show how the non-perturbatively generated Affleck-Dine-Seiberg superpotential can be obtained by the fluxes of the dual supergravity solution together with some geometric considerations.

2. SQCD from fractional branes on $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$

The system we are going to consider is a stack of fractional D3-branes on the orbifold $\mathbb{R}^{1,3} \times \mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ [19]. In this space, which is a singular and non-compact Calabi-Yau three-fold, a stack of D-branes will preserve four supercharges. In particular we arrange the fractional branes according to table 1, where $-$ and $\cdot$ indicate, respectively, longitudinal and transverse directions. In the following we will denote with x^α ($\alpha = 0, \ldots, 3$) the coordinates transverse to the orbifold, and introduce three complex coordinates in the orbifolded directions x^r ($r = 4, \ldots, 9$) as follows

					$\overbrace{\qquad\qquad}^{\mathbb{C}^3/\mathbb{Z}_2\times\mathbb{Z}_2}$					
	0	1	2	3	4	5	6	7	8	9
D3	–	–	–	–	·	·	·	·	·	·

Table 1: Fractional D3-branes on $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$.

$$z_1 = x^4 + ix^5\,, \qquad z_2 = x^6 + ix^7\,, \qquad z_3 = x^8 + ix^9\,. \tag{2.1}$$

The generators of the two $\mathbb{Z}_2$ factors of the orbifold group are denoted by g_1 and g_2, and their action on the complex coordinates is given by

$$\begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline g_1 & z_1 & -z_2 & -z_3 \\ g_2 & -z_1 & z_2 & -z_3 \end{array} \tag{2.2}$$

The remaining two elements of the orbifold group are of course the identity e and $g_3 = g_1 g_2$.

As is well known, the most elementary configurations of branes on orbifolds are made by fractional branes [20]. These are defined by the fact that the Chan-Paton factors of the open strings attached to them transform in the irreducible representations of the orbifold group; moreover, they have the property of being stuck at the orbifold fixed point. In our $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, we then have four different types of fractional D3-branes, that we denote as A, B, C, D, corresponding to the four irreducible one-dimensional representations of the orbifold

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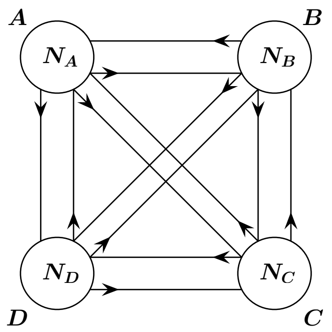

Figure 1: Each node i of the quiver diagram represents a gauge group factor $\mathrm{U}(N_i)$ with the corresponding vector multiplet, and each oriented arrow from node i to node j represents a chiral multiplet in the $(\mathbf{N_i}, \bar{\mathbf{N}}_\mathbf{j})$ representation.

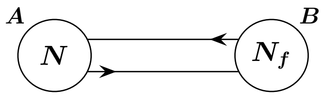

Figure 2: The quiver diagram associated to a system of N fractional D3-branes of type A and N_f fractional D3-branes of type B which is used to engineer $\mathrm{U}(N)$ SQCD with N_f flavors.

group. The low-energy theory living on a generic system of N_k D3-branes of type k, is a four-dimensional $\mathcal{N} = 1$ gauge theory with gauge group $\mathrm{U}(N_\mathrm{A}) \times \mathrm{U}(N_\mathrm{B}) \times \mathrm{U}(N_\mathrm{C}) \times \mathrm{U}(N_\mathrm{D})$ and twelve chiral multiplets, which transform in the fundamental (or anti-fundamental) representation of a particular gauge group factor and carry a flavor index with respect to the other three factors. All this information can be encoded in the quiver diagram represented in figure 1.

Our goal is to use this orbifold set-up to engineer $\mathrm{U}(N)$ $\mathcal{N} = 1$ SQCD, which is a theory with N_f flavors of "quarks" and "antiquarks". A possibility to do this would be to consider a stack of N fractional D3-branes of a given type, and add to them N_f D7-branes in order to introduce the fundamental and anti-fundamental chiral multiplets [8]. Here, however, we will follow an alternative and simpler route [21], which allows to obtain the main features of $\mathcal{N} = 1$ SQCD in a very natural way by using only D3-branes. This alternative strategy amounts simply to consider a configuration of N fractional D3-branes of, say, type A and N_f fractional D3-branes of, say, type B, which gives rise to the $\mathrm{U}(N) \times \mathrm{U}(N_f)$ gauge theory represented by the diagram of figure 2.

From this quiver model we clearly can obtain $\mathrm{U}(N)$ SQCD with N_f flavors if we concentrate only on the low-energy degrees of freedom associated to the branes of type A by a suitable selection of open strings with appropriate Chan-Paton factors. Even if in the complete theory the flavor symmetry $\mathrm{U}(N_f)$ is also gauged, we will see that this configuration of fractional D3 branes is able to provide a big deal of information on SQCD via the gauge/gravity correspondence.

3. Summary of the supergravity solution

The classical supergravity solution describing a system of N_k fractional D3-branes of type k placed at the origin $z_1 = z_2 = z_3 = 0$ of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ was constructed in Ref. [7]. Here we briefly summarize its main properties.

Let us first recall that the bosonic fields of type IIB supergravity are the metric, the 2-form B_2 and the dilaton ϕ from the NS-NS sector, and the 0-form C_0, the 2-form C_2 and the 4-form C_4 with self-dual field strength $\tilde{F}_5$ from the R-R sector. Then, the fractional D3-brane solution is obtained by assuming that the two 2-forms B_2 and C_2 have components only along the three exceptional shrinking 2-cycles $\mathcal{C}_i$ $(i = 1, 2, 3)$ of the orbifold geometry. Specifically, one writes

$$B_2 = b_i\ \omega_2^{(i)}\ , \qquad C_2 = c_i\ \omega_2^{(i)}\ , \tag{3.1}$$

where the anti-self dual (1,1)-forms $\omega_2^{(i)}$, dual to the 2-cycles $\mathcal{C}_i$, are defined and normalized as

$$\int_{\mathcal{C}_i} \omega_2^{(j)} = \delta_i^j\ , \qquad \int \omega_2^{(i)} \wedge \omega_2^{(i)} = -\frac{1}{4}\ , \qquad {}^{\star_4}\omega_2^{(i)} = -\omega_2^{(i)}\ . \tag{3.2}$$

The scalar fields b_i and c_i in (3.1) are called twisted fields, and precisely correspond to massless degrees of freedom of the three NS-NS and R-R twisted sectors of the conformal field theory describing closed strings in the orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$.

If we define

$$G_3 = dC_2 + \big(C_0 + \mathrm{i}\,\mathrm{e}^{-\phi}\big) dB_2\ , \tag{3.3}$$

and

$$\gamma_i = c_i + \mathrm{i}\, b_i\ , \tag{3.4}$$

then the solution for a stack of N_k fractional D3 branes of type k reads [7]

$$\begin{aligned} ds^2 &= H_3^{-1/2}\, \eta_{\alpha\beta}\, dx^\alpha dx^\beta \ +\ H_3^{1/2}\, \delta_{rs}\, dx^r dx^s\ , \\ C_0 &= 0\ , \qquad \mathrm{e}^\phi = 1\ , \\ \tilde{F}_5 &= dH_3^{-1}\, dx^0 \wedge \cdots \wedge dx^3 + {}^\star \big(dH_3^{-1}\, dx^0 \wedge \cdots \wedge dx^3\big)\ , \\ G_3 &= d\gamma_i \wedge \omega_2^{(i)}\ . \end{aligned} \tag{3.5}$$

Here the functions γ_i are given by

$$\gamma_i = \mathrm{i}\, K\, f_i(N_k)\ \ln \frac{z_i}{\epsilon_0}\ , \tag{3.6}$$

where ϵ_0 is a short-distance regulator, $K = 4\pi g_s {l_s}^2$ (g_s being the string coupling constant and l_s the string length) and the f_i's are functions of the numbers of the different types of fractional branes

$$\begin{aligned} f_1(N_k) &= N_{\mathrm{A}} + N_{\mathrm{B}} - N_{\mathrm{C}} - N_{\mathrm{D}}\ , \\ f_2(N_k) &= N_{\mathrm{A}} - N_{\mathrm{B}} + N_{\mathrm{C}} - N_{\mathrm{D}}\ , \\ f_3(N_k) &= N_{\mathrm{A}} - N_{\mathrm{B}} - N_{\mathrm{C}} + N_{\mathrm{D}}\ . \end{aligned} \tag{3.7}$$

Finally, H_3 is a specific function of z_i whose explicit expression, which we will not need in the following, can be found in ref. [7].

As is clear from the solution (3.5), each individual fractional D3-brane is charged under all three twisted sectors of the closed string theory on the orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$, as well as under the untwisted one. This is perfectly consistent with the description of fractional D-branes in terms of boundary states which represent the sources for all closed string states emitted by the D-branes. In our specific case, the boundary states for the various fractional

D3-branes are schematically given by

$$\begin{aligned}
|\mathrm{A}\rangle &= |\mathrm{U}\rangle + |\mathrm{T1}\rangle + |\mathrm{T2}\rangle + |\mathrm{T3}\rangle\,, \\
|\mathrm{B}\rangle &= |\mathrm{U}\rangle + |\mathrm{T1}\rangle - |\mathrm{T2}\rangle - |\mathrm{T3}\rangle\,, \\
|\mathrm{C}\rangle &= |\mathrm{U}\rangle - |\mathrm{T1}\rangle + |\mathrm{T2}\rangle - |\mathrm{T3}\rangle\,, \\
|\mathrm{D}\rangle &= |\mathrm{U}\rangle - |\mathrm{T1}\rangle - |\mathrm{T2}\rangle + |\mathrm{T3}\rangle\,,
\end{aligned} \tag{3.8}$$

where $|U\rangle$ is the contribution of the untwisted sector, $|\mathrm{T}i\rangle$ is the one of the i-th twisted sector corresponding to the group element g_i, and the signs are consistent with (3.7).

Finally, we remark that the supergravity solution (3.5) has a naked singularity of repulson type, a common feature of all classical solutions describing fractional branes on orbifolds. One should then proceed to examine the appearance of an enhançon mechanism [22] that would make this geometry acceptable, but we will not perform this analysis here. Instead, in the following sections we will use the above solution to study the dual $\mathcal{N}=1$ gauge theory.

4. The dual gauge theory and its classical moduli space

Let us now concentrate on the configuration made up of N fractional D3-branes of type A and N_f fractional D3-branes of type B that we introduced in section 2 and represented in figure 2. As we have seen, the theory on the world-volume of the type A branes is U(N) SQCD with $\mathcal{N}=1$ supersymmetry, N_f "quark" chiral multiplets, Q^i, and N_f "antiquark" chiral multiplets, $\tilde{Q}_{\tilde{j}}$.

Many properties of this gauge theory can be explicitly obtained from the supergravity solution (3.5) (with $N_C = N_D = 0$). As a first example, let us consider the running gauge coupling constant g_{YM}, which, as shown in ref. [7], can be directly related to the twisted scalars b_i of the dual geometry, according to

$$\frac{1}{g^2_{\mathrm{YM}}} = \frac{1}{8\pi g_s}\frac{1}{(2\pi l_s)^2}\sum_{i=1}^{3} b_i\,. \tag{4.1}$$

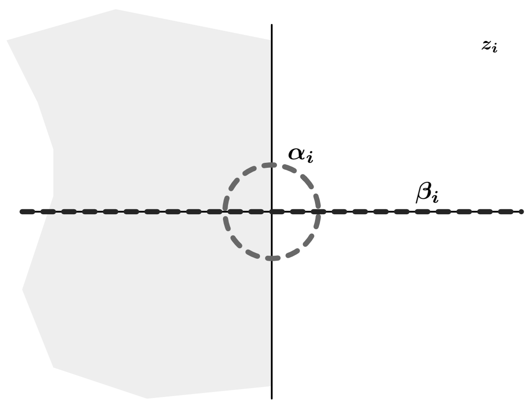

Figure 3: The compact 1-cycle α_i and the non-compact 1-cycle β_i in the z_i plane.

The right hand side of this equation can be equivalently written also in terms of the flux of G_3 across an appropriate singular 3-cycle of the orbifold $\mathbb{C}^3/\mathbb{Z}_2\times\mathbb{Z}_2$. To see this, and also for our later analysis of the superpotential, it is useful to identify the singular 3-cycles that exist in this non-compact Calabi-Yau space.

Since there are no exceptional (1,2) or (2,1)-forms coming from the twisted sectors, everything should arise from the (1,1)-forms $\omega_2^{(i)}$, dual to the singular 2-cycles $\mathcal{C}_i$ that we already encountered in section 3. Thus, we can introduce three compact 3-cycles A_i and three non-compact 3-cycles B_i by simply taking the direct product of $\mathcal{C}_i$ with suitable 1-cycles in the z_i planes. Specifically, we define

$$A_i = \alpha_i \times \mathcal{C}_i\,, \qquad B_i = \beta_i \times \mathcal{C}_i \qquad (i = 1, 2, 3)\,, \tag{4.2}$$

where the compact cycles α_i and the non-compact cycles β_i in the z_i plane are orthogonal to each other and are shown in figure 3.

Using the last equation of (3.5) and the normalizations (3.2), one can easily see that the fluxes of G_3 along these 3-cycles are given by

$$\int_{A_i\,(\text{ or } B_i)} G_3 = \int_{A_i\,(\text{ or } B_i)} d\gamma_j \wedge \omega_2^{(j)} = \int_{\alpha_i\,(\text{ or } \beta_i)} d\gamma_i \int_{\mathcal{C}_i} \omega_2^{(i)} = \int_{\alpha_i\,(\text{ or } \beta_i)} d\gamma_i\,. \tag{4.3}$$

In particular, for our specific brane configuration, we find

$$\int_{A_1} G_3 = -2\pi K(N + N_f)\,, \qquad \int_{A_2} G_3 = \int_{A_3} G_3 = -2\pi K(N - N_f)\,, \tag{4.4}$$

and

$$\int_{B_1} G_3 = \mathrm{i}\,K(N + N_f)\,\ln\frac{\rho_c}{\rho_0}\,, \qquad \int_{B_2} G_3 = \int_{B_3} G_3 = \mathrm{i}\,K(N - N_f)\,\ln\frac{\rho_c}{\rho_0}\,. \tag{4.5}$$

In the last line, the integrations over $\rho_i = |z_i|$ in the non-compact cycles β_i extend up to a cut-off ρ_c, which sets the higher scale, while the lower scale ρ_0 is introduced as a further short-distance cut-off, since the validity of the singular supergravity solution stops at a finite distance from the brane position.

If we identify ρ_0 with the (arbitrary) cut-off ϵ_0 appearing in (3.6), we immediately see that $b_i = -\mathrm{i}\int_{B_i} G_3$, and thus (4.1) becomes

$$\frac{1}{g^2_{\mathrm{YM}}} = \frac{1}{8\pi g_s}\frac{1}{(2\pi l_s)^2}\sum_{i=1}^{3}\left(-\mathrm{i}\int_{B_i} G_3\right) = \frac{1}{8\pi^2}\,(3N - N_f)\ln\frac{\rho_c}{\rho_0}\,. \tag{4.6}$$

Introducing the renormalization scale μ and the dynamically generated scale Λ through the usual energy/radius relations

$$\rho_c = 2\pi {l_s}^2\,\mu\,, \qquad \rho_0 = 2\pi {l_s}^2\,\Lambda\,, \tag{4.7}$$

that follow from a "stretched string" analysis [23], we easily get from (4.6) the correct one-loop running coupling constant for our U(N) SQCD theory.

We now turn to the vacuum structure of this theory and show how it can be recovered from the supergravity solution (3.5) of fractional D3-branes in a very simple way. Since it is known that U(N) SQCD has a very different behavior depending on the number of flavors (see for instance Ref. [24]), here we concentrate on the case in which the number of flavors N_f is less than the number of colors N. This is the so-called Affleck-Dine-Seiberg (ADS) theory [18], in which an effective superpotential is generated by non-perturbative effects.

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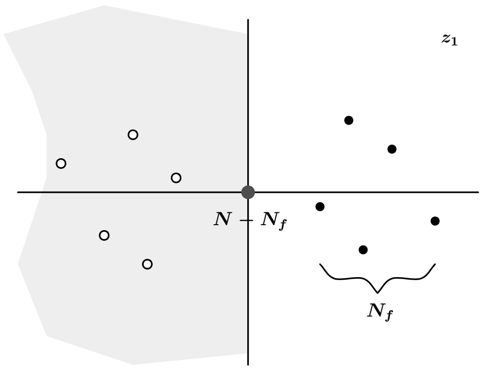

Figure 4: Moduli space of the ADS theory via fractional branes: A+B superpositions at arbitrary points of the z_1 plane, together with their images on the covering space.

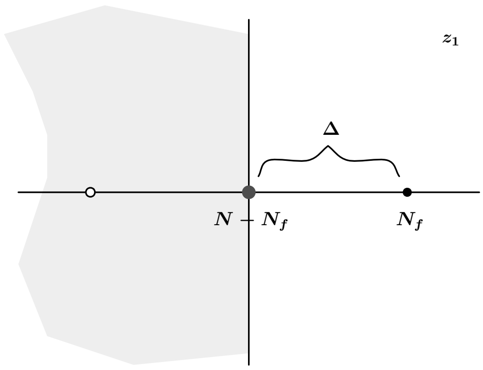

Figure 5: Moduli space of the ADS theory via fractional branes: The configuration which makes the meson matrix $M^i_{\tilde{j}}$ proportional to the identity.

To see how the vacuum structure of the ADS theory can be recovered from supergravity, let us begin by considering a single fractional D3-brane of type A together with one of type B. As individual objects, these branes are charged under all four sectors of closed string theory, and for this reason they cannot move off the orbifold fixed point (see the corresponding boundary states (3.8)). However, since A and B are mutually BPS objects, we can easily construct the superposition A + B, which will be charged under the sector twisted by g_1 (with a charge double with respect to a single fractional brane), but will not carry any charge under the other two twisted sectors. This can be easily seen in the boundary state description, where one writes

$$|\mathrm{A}+\mathrm{B}\rangle = |\mathrm{A}\rangle + |\mathrm{B}\rangle = 2\,\left(|\mathrm{U}\rangle + |\mathrm{T1}\rangle\right). \tag{4.8}$$

Thus, the superposition A+B can freely move in the z_1 plane, which is left fixed by the action of g_1. Notice that when the pair A+B is not at the origin, it cannot be divided anymore into its components, which indeed are defined only at the orbifold fixed point. The crucial observation is that the motion of A+B causes the breaking of the $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge group of the theory living on the superposition down to $\mathrm{U}(1)$ via the Higgs mechanism.

Returning to our SQCD configuration and repeating the above argument, one finds that out of the N branes of type A and the N_f branes of type B, it is possible to build N_f A + B superpositions and move them away from the origin at arbitrary points in the z_1 plane. The motion of these A + B pairs is naturally interpreted as giving arbitrary vacuum expectation values to the meson matrix $M^i_{\tilde{j}} = Q^i\tilde{Q}_{\tilde{j}}$, thus breaking the gauge group $\mathrm{U}(N)$ down to $\mathrm{U}(N-N_f)$ corresponding to the world-volume theory of remaining $N-N_f$ fractional branes of type A still placed at the origin. Therefore, this D-brane construction uncovers the correct classical moduli space of the ADS theory in a very natural way, as shown in figure 4.

This same mechanism works also for $N = N_f$. In this case, all branes can form pairs and, since no unpaired A branes are left, no unbroken gauge theory remains. Notice that this description makes very clear that something drastic happens when passing from

$N_f \leq N$ to $N_f > N$, in agreement with the known field theory results [24]. Indeed, if $N_f > N$, all we have done is no longer valid and one is forced to look for some alternative description.

Let us now discuss how the supergravity solution (3.5) gets modified when we form the A + B superpositions and move them in the z_1 plane. For simplicity, but without any loss of generality, we place all N_f superpositions at the same point $z_1 = \Delta$ on the real axis of the z_1 plane, as shown in figure 5 (clearly, in order to have an even configuration under the orbifold group, we also need to put images at the identified point $z_1 = -\Delta$). This set-up makes the meson matrix proportional to the identity, $M^i_j = m^2 \delta^i_j$, where m is related to Δ via the usual gauge/gravity identification

$$\Delta = 2\pi {l_s}^2 \, m\,. \tag{4.9}$$

We will be interested in the case in which Δ and m are very large.

As we have seen before, what is relevant for the dual gauge theory is the knowledge of the twisted fields γ_i, which now become

$$\gamma_i = \mathrm{i}\, K \left[(N - N_f) \ln \frac{z_i}{\epsilon_0} + \delta_{i,1}\; N_f \;\ln \frac{z_1 - \Delta}{\epsilon_0} + \delta_{i,1}\; N_f \;\ln \frac{z_1 + \Delta}{\epsilon_0} \right]. \tag{4.10}$$

Then, using (4.2) and (4.3), we find that the flux of G_3 along the A_i cycles is given by

$$\int_{A_i} G_3 = -2\pi K (N - N_f) \tag{4.11}$$

for all i's, while the flux along the B_i cycles becomes

$$\int_{B_i} G_3 = \mathrm{i}\, K \left[(N - N_f) \ln \frac{\rho_c}{\rho_0} + \delta_{i,1}\; 2N_f \;\ln \frac{\Delta}{\rho_0} \right], \tag{4.12}$$

where we have assumed $\rho_c \ll \Delta$ and denoted again by ρ_0 the minimum distance (from the branes at the origin and the pairs in $\pm\Delta$) that one can reach along the integration path on $\rho_i = |z_i|$ in the β_i cycles.[1]

Thus, the gauge coupling constant for the world-volume theory of the unpaired A branes is

$$\frac{1}{g^2_{\mathrm{YM}}} = \frac{1}{8\pi g_s} \frac{1}{(2\pi l_s)^2} \sum_{i=1}^{3} \left(-\mathrm{i} \int_{B_i} G_3 \right) = \frac{3}{8\pi^2} (N - N_f) \ln \frac{\rho_c}{\rho_0} + \frac{2N_f}{8\pi^2} \ln \frac{\Delta}{\rho_0}\,, \tag{4.13}$$

which in terms of the gauge theory scales becomes

$$\frac{1}{g^2_{\mathrm{YM}}} = \frac{3}{8\pi^2} (N - N_f) \ln \frac{\mu}{\Lambda} + \frac{2N_f}{8\pi^2} \ln \frac{m}{\Lambda}\,, \tag{4.14}$$

[1] Notice that if instead $\rho_c \gg \Delta$, the flux of G_3 along the B_i cycles is given by (4.5), in agreement with the fact that in this case all degrees of freedom are light and thus contribute to the running of the coupling constant as in (4.6).

with $\mu \ll m$. The above coupling can be expressed also in terms of the low-energy effective unbroken $\mathrm{U}(N - N_f)$ theory as

$$\frac{1}{g_{\mathrm{YM}}^2} = \frac{3}{8\pi^2} (N - N_f) \ln \frac{\mu}{\Lambda_L} , \tag{4.15}$$

where the low-energy scale Λ_L is related to Λ and m via

$$\Lambda_L{}^{3(N-N_f)} = \frac{\Lambda^{3N-N_f}}{m^{2N_f}} = \frac{\Lambda^{3N-N_f}}{\det M} . \tag{4.16}$$

We then see that the supergravity computation precisely reproduces, beyond the running coupling constant, also the expected matching of scales in the gauge theory [24]. This concludes our analysis of the theory at the classical and perturbative levels. In the next section, we will turn to non-perturbative phenomena, with the computation of the effective superpotential.

5. The ADS superpotential

We now study the quantum moduli space of SQCD using our D-brane set-up, and see that our quantitative results perfectly agree with the field theory analysis, predicting that the classical moduli space is lifted by the generation of the ADS superpotential [18].

The main ingredient we will implement is the formula [17, 14] that, in the cases where the dilaton is constant, expresses the $\mathcal{N} = 1$ effective gauge superpotential W_{eff} in terms of the fluxes of the complex three-form G_3 of the dual IIB supergravity solution and the periods of the holomorphic (3,0)-form Ω of the Calabi-Yau orbifold in which the branes are put, namely

$$W_{\mathrm{eff}} \propto \sum_i \left[\int_{A_i} G_3 \int_{B_i} \Omega - \int_{A_i} \Omega \int_{B_i} G_3 \right] . \tag{5.1}$$

As a brief summary, we recall that this formula has been shown to give the correct answer in many set-ups where the gauge theory is engineered via D-brane configurations on Calabi-Yau manifolds which undergo a geometric transition, where some 3-cycles of the manifold blow up and the branes are replaced by fluxes of IIB supergravity fields [13, 14]. For example, using this approach the Veneziano-Yankielowicz superpotential [25] of pure $\mathcal{N} = 1$ super Yang-Mills theory has been extracted [15, 26] from the warped deformed conifold solution by Klebanov and Strassler [3].

In the case at hand, the supergravity solution we have at our disposal is not smooth, unlike the solution of Ref. [3]. Rather, we are in a situation, the orbifold limit, where all 2- and 3-cycles are shrinking, similarly to what happens in the singular conifold solution of Klebanov and Tseytlin [2]. However, this does not seem to be an obstacle for using (5.1). In fact, in the conifold case the knowledge of the singular Klebanov-Tseytlin solution is sufficient for computing the G_3 fluxes that are needed in (5.1), for the very simple reason that they are precisely identified with the fluxes of the regular Klebanov-Strassler solution.[2]

[2]This identification also explains the relation between the radial coordinate r appearing in the singular KT solution, and the coordinate τ appearing in the regular KS solution (see for example [11, eq. (100)]).

The same is true also in our case, and thus even if we do not know a regular solution in a deformed orbifold, the G_3 fluxes can be obtained from the singular solution as described in section 4. On the other hand, the periods of Ω that enter in (5.1) cannot be determined in the singular case, since they crucially depend on the details of the deformation of the geometry. However, the geometric considerations which are necessary for getting the correct periods do not depend on the details of the classical solution and can be worked out in full generality also for the orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, we can fruitfully combine the knowledge of the fluxes coming from our explicit supergravity solution with the geometric features of the deformed background. This is the approach we take with our fractional D-branes in the orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$.

We already derived the fluxes of G_3 along the A_i and B_i cycles: they are given in (4.11) and (4.12). Let us then consider the periods of the holomorphic $(3,0)$-form Ω. As in the case of the conifold, in order to get sensible results it is necessary to deform the singular geometry of the orbifold. Let us start by noting that the space $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ can be described as the $F(x,y,z,t) = 0$ hypersurface in $\mathbb{C}^4$, where

$$F(x,y,z,t) = xyz + t^2 \,. \tag{5.2}$$

The invariant variables in this function are related to the complex coordinates (2.1) by

$$x = z_1^2\,, \qquad y = z_2^2\,, \qquad z = z_3^2\,, \qquad t = \mathrm{i}\, z_1 z_2 z_3\,, \tag{5.3}$$

and thus their engineering dimensions are $[x] = [y] = [z] = L^2$, and $[t] = L^3$. The simplest deformation of the complex structure, which also resolves completely the singularity, is a constant deformation with parameter ξ, namely

$$F(x,y,z,t) \quad \to \quad F_\xi(x,y,z,t) = xyz + t^2 - \xi^2 \tag{5.4}$$

(notice that $[\xi] = L^3$). In ref. [21], Berenstein has shown, via holomorphy considerations strengthened by a matrix model computation, that (5.4) is indeed the correct deformation to consider. He also showed that the deformation parameter ξ is related to the gaugino condensate S of the dual gauge theory, as we too will argue below. Given (5.4), we can write the holomorphic (3,0)-form Ω in the usual way, according to

$$\Omega = \frac{1}{2\pi \mathrm{i}} \oint_{F_\xi = 0} \frac{dx \wedge dy \wedge dz \wedge dt}{F_\xi} = \frac{dx \wedge dy \wedge dz}{2\sqrt{\xi^2 - xyz}}\,. \tag{5.5}$$

In order to compute the periods of Ω along a specific A_i (or B_i) cycle, we define (with a little abuse of notation) $x = z_i^2$, $y = u + \mathrm{i}v$, $z = u - \mathrm{i}v$ and $\varepsilon^2 = \xi^2/x$, so that from (5.5) we get

$$\int \Omega = -\mathrm{i} \int \frac{dx \wedge du \wedge dv}{\sqrt{\xi^2 - x(u^2+v^2)}} = -\mathrm{i} \int \frac{dx}{\sqrt{x}} \int_{\mathcal{C}_i} \frac{du \wedge dv}{\sqrt{\varepsilon^2 - u^2 - v^2}}\,. \tag{5.6}$$

The last integral can be easily evaluated, and is in fact a well-known result in the context of the geometry of the K3 manifold

$$\int_{\mathcal{C}_i} \frac{du \wedge dv}{\sqrt{\varepsilon^2 - u^2 - v^2}} = \int_{-\varepsilon}^{\varepsilon} du \int_{\gamma_u} \frac{dv}{\sqrt{\varepsilon^2 - u^2 - v^2}} = \int_{-\varepsilon}^{\varepsilon} du \int_{\gamma_\infty} \frac{dw}{\mathrm{i}\, w} = 4\pi\varepsilon\,.$$

Using this inside (5.6), we then have

$$\int \Omega = -\mathrm{i} \int \frac{dx}{\sqrt{x}}\, 4\pi\, \frac{\xi}{\sqrt{x}} = -4\pi \mathrm{i}\, \xi \int \frac{dx}{x} = -8\pi \mathrm{i}\, \xi \int \frac{dz_i}{z_i} \,. \tag{5.7}$$

Thus, the periods of Ω along the cycles A_i and B_i are finally given by

$$\int_{A_i} \Omega = -8\pi \mathrm{i}\, \xi \oint \frac{dz_i}{z_i} = 16\pi^2 \xi \,, \tag{5.8}$$

and

$$\int_{B_i} \Omega = -8\pi \mathrm{i}\, \xi \int_{\xi^{1/3}}^{\rho_c} \frac{d\rho_i}{\rho_i} = \frac{8\pi \mathrm{i}}{3}\, \xi\, \ln \frac{\xi}{\rho_c^3} \,, \tag{5.9}$$

where in the latter we have used the same upper cutoff ρ_c already used in the computation of the fluxes of G_3, while the lower limit of integration must now be given by a suitable power of the deformation parameter ξ (notice that since ξ has dimension L^3, this power is $1/3$ in order to match with the length dimension of $\rho_i = |z_i|$).

We have now all the necessary ingredients to compute the effective superpotential of the gauge theory by means of formula (5.1). Inserting the appropriate dimensionful prefactors, we find that the gauge effective superpotential is given by

$$\begin{aligned} W_{\text{eff}} &= \frac{1}{16\pi^2 \mathrm{i}\, K}\, \frac{1}{(2\pi {l_s}^2)^3} \sum_{i=1}^{3} \left[\int_{A_i} G_3 \int_{B_i} \Omega - \int_{A_i} \Omega \int_{B_i} G_3 \right] \\ &= -\frac{1}{(2\pi {l_s}^2)^3} \left[3(N-N_f)\, \frac{\xi}{3}\, \ln \frac{\xi}{\rho_c^3} + 3(N-N_f)\, \xi\, \ln \frac{\rho_c}{\rho_0} + 2N_f\, \xi\, \ln \frac{\Delta}{\rho_0} \right] \end{aligned} \tag{5.10}$$

We now re-express the geometrical quantities in terms of gauge theory quantities, by using again the "stretched string" energy/radius relation. Notice that the deformation parameter ξ, due to its engineering dimensions, is identified by the relation with a mass dimension 3 operator in the gauge theory, which is then natural to identify with the gaugino condensate S (see also ref. [21]). In summary the gauge/gravity relations are

$$\rho_c = 2\pi {l_s}^2\, \mu \,, \qquad \rho_0 = 2\pi {l_s}^2\, \Lambda \,, \qquad \Delta = 2\pi {l_s}^2\, m \,, \qquad \xi = (2\pi {l_s}^2)^3\, S \,, \tag{5.11}$$

so that (5.10) becomes

$$W_{\text{eff}} = -(N-N_f)\, S\, \ln \frac{S}{\Lambda^3} - 2N_f\, S\, \ln \frac{m}{\Lambda} \,. \tag{5.12}$$

Though this result is correct, let us redefine the scales in order to write it in a more conventional way. The appropriate redefinition is $\Lambda \to e^{1/3}\Lambda$, $m \to e^{1/3} m$, and we get

$$W_{\text{eff}} = (N-N_f) \left[S - S\, \ln \frac{S}{\Lambda^3} \right] - 2N_f\, S\, \ln \frac{m}{\Lambda} \,, \tag{5.13}$$

which is precisely the Taylor-Veneziano-Yankielowicz superpotential for SQCD with N_f flavors [27]. At the minimum the gaugino condensate is

$$S = \left(\frac{\Lambda^{3N-N_f}}{m^{2N_f}} \right)^{\frac{1}{N-N_f}} \,, \tag{5.14}$$

so that from (5.13) we get the ADS superpotential [18]:

$$W_{\text{eff}} = (N - N_f) \left[\frac{\Lambda^{3N-N_f}}{m^{2N_f}} \right]^{\frac{1}{N-N_f}} = (N - N_f) \left[\frac{\Lambda^{3N-N_f}}{\det M} \right]^{\frac{1}{N-N_f}} . \tag{5.15}$$

As an aside, we note that if $N_f = 0$ the above results reproduce the Veneziano-Yankielowicz superpotential for pure $\mathcal{N} = 1$ Super Yang-Mills theory [25]

$$W_{\text{VY}} = N \left[S - S \ln \frac{S}{\Lambda^3} \right] . \tag{5.16}$$

Its value at the minimum (where $S = \Lambda^3$) is $W_{\text{VY}} = N\Lambda^3$. We stress that this is not just a formal limit of the result obtained for $N_f > 0$; indeed, one could have started from the beginning by considering only N fractional branes of type A, and apply formula (5.1) to obtain the Veneziano-Yankielowicz superpotential.

Another observation concerns the case in which $N_f = N$. As we have mentioned before, our brane construction can still be used in this case where all branes form A + B superpositions and no fractional branes (and thus no effective gauge theory) are left. Therefore, the result for the moduli space can be read from the superpotential (5.13) for $N_f = N$. In this case, the minimization procedure implies $\det M = \Lambda^{2N}$, which is indeed the correct result expected for a $\mathcal{N} = 1$ gauge theory with gauge group $\mathrm{U}(N)$ and N flavors.

Therefore we can conclude that our classical supergravity solution, together with some geometrical considerations, has been able to provide relevant information on the $\mathcal{N} = 1$ SQCD with N_f flavors, both at at the classical and perturbative level, and also at a non-perturbative level. It would be very interesting to use this system of fractional branes to analyze SQCD also in the phase where $N_f > N$, where Seiberg duality is supposed to take place [28]. Indeed a construction of Seiberg duality for generic quiver theories was presented in Refs. [29, 21] with the implementation of quite formal methods involving also anti-branes. A more explicit analysis of these methods in the specific model we have studied here is under current investigation [30].

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JHEP12(2003)051

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